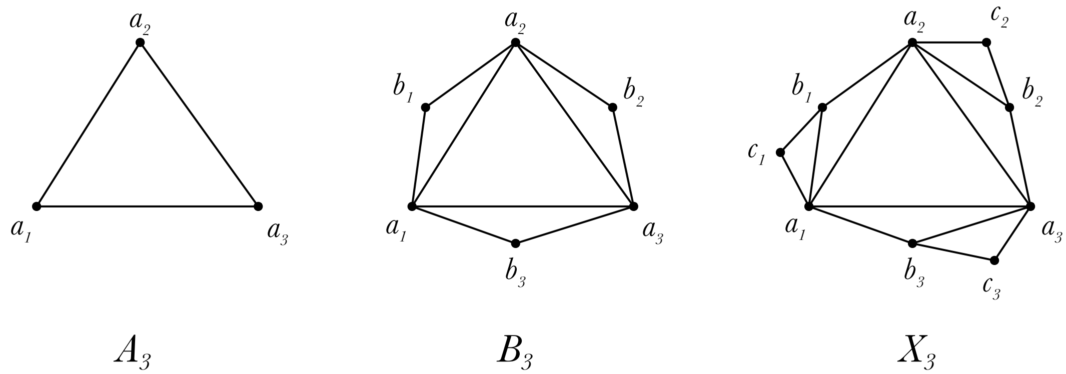
SOME CYCLIC-AUTOMORPHIC GRAPHS

Let n be a natural number no less than 3, and let C_n be the cyclic group of order n. We will construct a simple, undirected graph X_n whose group of symmetries is C_n, that is, a graph X_n with $Aut(X_n) \cong C_n$.



Begin with the cyclic graph of order n, with labelled vertices $a_1, \ldots, a_n$ and edges $e_1, \ldots, e_n$. Let this graph be A_n. Let B_n grow from A_n with the addition of n vertices $b_1, \ldots, b_n$ and $2n$ edges $f_1, \ldots, f_{2n}$ such that for every edge e_i of A_n, in B_n there is path of length two connecting the endpoints of e_i via vertex b_i.

Finally, create X_n from B_n by adding n more vertices $c_1, \ldots, c_n$ and $2n$ more edges such that for every even numbered edge f_{2i} in B_n, in X_n there is a path of length two connecting the endpoints of f_{2i} via vertex c_i.

Theorem 1. $Aut(X_n) \cong C_n$.

Proof. As labelled above, the vertices may be classified a_i with degree 5, b_i with degree 3, and c_i with degree 2. An automorphism must obviously preserve the degree of a vertex.

Let α be the permutation of vertices $(a_1 a_2 \ldots a_n)(b_1 b_2 \ldots b_n)(c_1 c_2 \ldots c_n)$ in disjoint cycle notation. Clearly α does not change the graph, and so $\langle \alpha \rangle \subseteq Aut(X_n)$.

Now let $\beta \in Aut(X_n)$. We show that $\beta \in \langle \alpha \rangle$. There are two cases to consider.

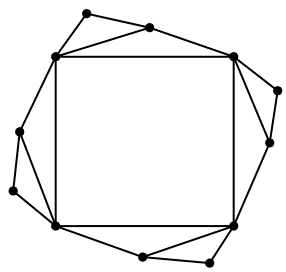
Case 1: β fixes c_1. As c_1 has only neighbors a_1 and b_1, of unequal degrees, β must leave a_1 and b_1 fixed. b_1 has only neighbors a_1 and c_1 and a_2, and since two of these are already known to be fixed, the third, a_2 must be fixed. Now, a_2 has the five neighbors sorted by degree: 5: a_1, a_3 , 3: b_1, b_2 , 2: c_2 . Thus c_2 is fixed and since a_1 is fixed a_3 must be fixed, and since b_1 is fixed b_2 must be fixed. Continuing around in this fashion we must eventually reach c_n and find all vertices fixed, and so $\beta = 1 \in \langle \alpha \rangle$.

Case 2: $\beta(c_1) \neq c_1$. The degree of $\beta(c_1)$ must be 2, thus $\beta(c_1) = c_j$ for some j. Then consider α^{j-1}. The product $\beta \cdot (\alpha^{j-1})^{-1}$ leaves c_1 fixed and by case 1 must be the neutral permutation. Thus $\beta = \alpha^{j-1}$.

□

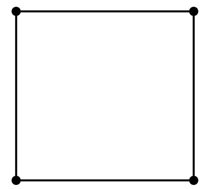
We call a graph such as X_n whose automorphism group is cyclic, *cyclic-automorphic*. Similary, $Cyclic_n$, the cyclic graph of order n is *dihedral-automorphic*, and the complete and empty graphs of order n, K_n and $\overline{K_n}$ are *symmetric-automorphic*. More generally for an abstract group G, we can say a graph is G-*automorphic* if its group of symmetries is isomorphic to G.

Adam Marks, December 14, 2011 v3 January 11, 2012.



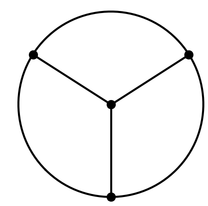
X_4

cyclic-automorphic



Cyclic$_4$

dihedral-automorphic



K_4

symmetric-automorphic

Having observed graphs whose automorphism groups are these common families of groups, the question may naturally be raised: for any finite group G does there necessarily exist a G-automorphic graph X?

It turns out that this question was posed by Denes Konig in 1936, in the first textbook on graph theory, and answered in the affirmative by Robert Frucht in 1938. Subsequently in 1948, Frucht expanded his result, giving a construction of a *cubic* G-automorphic graph for any given finite abstract group G. The proof above is derived from that paper. Note that the above construction also provides an answer to a question in that paper as to the existence of a C_3-automorphic graph of order 9.

References

[1] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, North Holland, 1976. http://www.math.jussieu.fr/ jabondy/books/gtwa/gtwa.html

[2] Robert Frucht, Graphs of Degree Three with a Given Abstract Group, Canadian Journal of Mathematics 1 (1949) 365-378. http://cms.math.ca/cjm/v1/p365

[3] Wikipedia contributors, Frucht's theorem, Wikipedia, The Free Encyclopedia, Accessed July 25, 2011.